Model-Following in Linear-Quadratic Optimization

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The use of models (linear differential equations embodying the desired properties) in the design of linear feedback systems via linear-quadratic optimization is treated. Two methods are discussed and compared on a numerical, lateral-axis aircraft control application: explicit model-following, which entails the simulation of the model as part of the feedforward controller, and implicit model-following, where the model serves to define the performance index. By and large, the implicit model method is found to be superior over explicit model-following. For implicit model-following, it is shown that 1) a cross-product term introduced into the performance index vanishes for several classes of systems, including those of flight-control; 2) one part of the control-law matrix *K* minimizes the integrand of the performance index and dominates *K* as the weight of the states in the index is increases; and 3) every desired linear control law can be obtained via optimal implicit model-following by a suitably defined model.

I. Introduction

In the usual linear-quardratic optimization, one is concerned with a linear, time-invariant, completely controllable plant described by a vector differential equation, connecting the state vector x and the control vector u:

$$\dot{x}=Ax+Bu \tag{1}$$

and with a performance index to be minimized:

$$I=\int_0^\infty (x^TQx+u^TRu)\,dt \tag{2}$$

where Q and R are constant, symmetric matrices, nonnegative definite and positive definite, respectively, and where superscript T denotes matrix transposition. As is well known, minimization of (2) results in a linear control law[1]

$$u=Kx \tag{3}$$

In a great many design problems, particularly in flight control, one would like the closed-loop system, given by (1) and (3),

$$\dot{x}=(A+BK)x \tag{4}$$

to be as close as possible to a system given by a differential equation

$$\dot{z}=A_m z \tag{5}$$

which represents the model of desirable dynamics, such as transient behavior, decoupling of modes, handling qualities of aircraft, etc. Since x and z may not be of the same dimension, one compares z with an output vector y of the same dimension, which is related to x by a matrix C:

$$y=Cx \tag{6}$$

In implicit model-following, the performance index is modified as

$$I_i=\int_0^\infty [(\dot{y}-A_m y)^T Q_i(\dot{y}-A_m y)+u^TRu]\,dt, \quad Q_i>0 \tag{7}$$

By substituting (1) and (6) for y and $\dot{y}$ in (7), this index becomes

$$I_i=\int_0^\infty (x^T\hat{Q}x+2u^T\hat{S}x+u^T\hat{R}u)\,dt \tag{8}$$

where

$$\hat{Q}=(CA-A_mC)^TQ_i(CA-A_mC),$$
$$\hat{S}=B^TC^TQ_i(CA-A_mC),\ \hat{R}=R+B^TC^TQ_iCB \tag{9}$$

Thus, the implicit model index (7) is seen in (8) to be equivalent to one of the standard type but with a cross-product term $u^T\hat{S}x$.

In explicit model-following, the index (2) is modified as

$$I_e=\int_0^\infty [(y-z)^TQ_e(y-z)+u^TRu]\,dt \tag{10}$$

so as to minimize the difference between the plant's output y and the model's state z. This index, too, can be transformed to the standard index (2). We define an augmented state ξ as

$$\dot{\xi}=\begin{bmatrix}\dot{x}\\ \dot{z}\end{bmatrix}=\begin{bmatrix}A & 0\\ 0 & A_m\end{bmatrix}\begin{bmatrix}x\\ z\end{bmatrix}+\begin{bmatrix}B\\ 0\end{bmatrix}u\triangleq\bar{A}\xi+\bar{B}u \tag{11}$$

and then (10) becomes

$$I=\int_0^\infty (\xi^T\bar{Q}\xi+u^TRu)\,dt \tag{12}$$

where

$$\bar{Q}=\begin{bmatrix}C^TQ_eC & -C^TQ_e\\ -Q_eC & Q_e\end{bmatrix} \tag{13}$$

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In contrast to implicit model-following, the model's equation must be simulated as part of a feedforward controller, because the optimal control now requires, in addition to the plant's state x, the model's state z:

$$u=\bar{K}\xi=K_1x+K_2z \tag{14}$$

Implicit model-following first was introduced in 1964 by Kalman et al.[2] but did not become as popular and well known as we think it deserves to be. This may be due to a number of factors. The method was downgraded on theoretical grounds, e.g., in Ref. 1, p. 86, because the cross-product term $u^T\hat{S}x$ prevents, in general, the closed-loop system from inherent possession of desirable properties such as a reduction of sensitivity to parameter variations. From a practical standpoint, explicit model-following was more acceptable, since it corresponds to the classical model-following and prefilter ideas. Furthermore, in a comparative study of the two methods,[3] Tyler favors explicit model-following as being "perhaps a more general method." In a later study,[4] Erzberger is of the opinion that the only disadvantage of explicit model-following is the added complexity, the need to incorporate the model in the controller, and that it will provide better model-following under conditions of unknown disturbances and parameter changes than implicit model-following. Consequently, most investigators[5-7] until quite recently[8,9] preferred explicit over implicit model-following. We note that the implicit model method as formulated in Refs. 2-4 is suitable for a regulator, and not a servo with command inputs. Thus, Tyler and Erzberger compared the two methods for responses to initial conditions only, assuming that the explicit model is excited by initial conditions identical to those in the actual plant; this accounts in great part for their conclusions.

In Sec. II, we formulate the implicit model method to include command inputs. In the process, we show that for many classes of systems the objectionable cross-product term vanishes [$\hat{S}=0$ and $\hat{R}=R$ in (9)], so that the implicit model in effect defines a nondiagonal full matrix Q in the standard index (2).

Section III develops properties of the control law resulting from the implicit model method. The two methods are discussed qualitatively in Sec. IV and are compared numerically in Sec. V on the basis of comparable gain magnitudes for an aircraft lateral-axis control design. Conclusions are given in Sec. VI.

II. Implicit Model-Following without a Cross-Product Term

We observe from (8) and (9) that, if B and C are such that

$$CB=0 \tag{15}$$

then

$$\hat{S}=0, \qquad \hat{R}=R \tag{16}$$

so that the index (7) reduces to the standard index (2), with Q replaced by a nondiagonal matrix $\hat{Q}$ defined in (9) by the difference between the plant and the model. The situation (15) occurs in many cases, for example, in a single-imput single-output system, where often

$$c^T=[1\,0...0] \qquad b^T=[0...0\,1]$$

A large and (in practice) important class of systems satisfying (15) is that of a main plant driven by an auxiliary plant. Typical examples are aircraft and ships where the control surfaces are moved by actuators, and where the inputs to these actuators are the control-surface commands. Such a system is shown in Fig. 1 and is described by the following:

Main Plant

$$\dot{x}_a=A_ax_a+B_a\delta \tag{17}$$

$$y=Cx_a \tag{18}$$

Auxiliary Plant

$$\dot{x}_\delta=A_\delta x_\delta+B_\delta u \tag{19}$$

$$\delta=C_\delta x_\delta \tag{20}$$

The command inputs δ_c are assumed to be generated by initial conditions on a vector differential equation that we call the command generator (see *Remark* at the end of this section):

$$\dot{x}_c=A_cx_c \tag{21}$$

$$\delta_c=C_cx_c \tag{22}$$

(Because of the application in Sec.V, we used aircraft notation here.) The entire system can be written in the form (1) by letting

$$x=\begin{bmatrix} x_a \\ x_\delta \\ x_c \end{bmatrix}, \quad A=\begin{bmatrix} A_a & B_aC_\delta & 0 \\ 0 & A_\delta & 0 \\ 0 & 0 & A_c \end{bmatrix}, \quad B=\begin{bmatrix} 0 \\ B_\delta \\ 0 \end{bmatrix} \tag{23}$$

We wish to design a control law, $u=K_1x_a+K_2x_\delta+K_3x_c$, so that the states x_a of the main plant behave like those of a model x_m given by

$$x_m=A_a'x_m+B_a'\delta_c \tag{24}$$

Here A_a' and B_a' are the desirable models for A_a amd B_a and are of the same dimensions, respectively. Adding the command generator, the entire model is given by (5), where now

$$z=\begin{bmatrix} x_m \\ x_c \end{bmatrix}, \qquad A_m\begin{bmatrix} A_a' & B_a'C_c \\ 0 & A_c \end{bmatrix} \tag{25}$$

Since the model (25) does not contain the actuators (considered a necessary evil), we define C in (18) as

$$C=\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \end{bmatrix} \tag{26}$$

It is obvious from (23) and (26) that (15) holds.

In the case (15), $\hat{Q}$ can be simplified by letting Q_i be of the block-diagonal form

$$Q_i=\begin{bmatrix} Q_a & 0 \\ 0 & Q_c \end{bmatrix}$$

where Q_a has the dimension of A_a, and Q_c the dimension of A_c. Then,

$$CA-A_mC=\begin{bmatrix} A_a-A_a' & B_a & -B_a' \\ 0 & 0 & 0 \end{bmatrix} \tag{27}$$

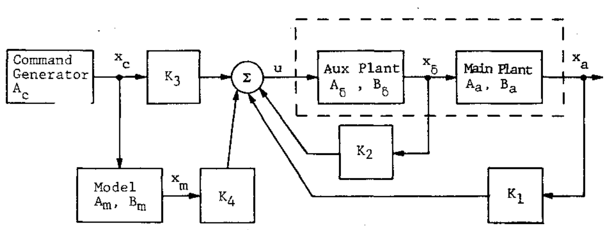

Fig. 1 Explicit model configuration.

and we have that $\hat{Q}$ in (9) is independent of Q_c:

$$\hat{Q}=[A_a-A_a' \; B_a-B_a']^T Q_a [A_a-A_a' \; B_a-B_a'] \tag{28}$$

The class of systems with $CB=0$ is, of course, not the only one where $\hat{S}=0$. For example, consider the case

$$A=\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, B=\begin{bmatrix} 0 \\ B_2 \end{bmatrix}, A_m=\begin{bmatrix} A_{m_{11}} & A_{m_{12}} \\ 0 & A_{m_{22}} \end{bmatrix} \tag{29}$$

where A_{11} and $A_{m_{11}}$ are of the same dimension and

$$A_{m_{22}}=A_{22} \tag{30}$$

Then, assuming that $C=I$,

$$A-A_m=\begin{bmatrix} A_{11}-A_{m_{11}} & A_{12}-A_{m_{12}} \\ 0 & 0 \end{bmatrix} \tag{31}$$

and, letting Q_i be in block-diagonal form,

$$Q_i=\begin{bmatrix} Q_{11} & 0 \\ 0 & Q_{22} \end{bmatrix} \tag{32}$$

we find that (9) becomes

$$\hat{Q}=[A_{11}-A_{m_{11}} \; A_{12}-A_{m_{12}}]^T Q_{11} [A_{11}-A_{m_{11}} \; A_{12}-A_{m_{12}}]$$

$$\hat{S}=0, \qquad \hat{R}=R+B_2^T Q_{22} B_2 \tag{33}$$

It is evident from the previous discussion that the possibility of a cross-product term in the index should not be considered a serious barrier to the use of the implicit model method. If the problem formulation involves the creation of a cross-product term $\hat{S}\neq 0$, then it can be eliminated by adding an arbitrarily fast actuator to precede the plant. Of course, the theoretically adverse effect of a cross-product term may, in practice, be negligibly small.

Remark. The use of a command generator converts the servo problem to a regulator problem; it is a standard way of handling command inputs (e.g., Ref. 1, Chap. 11) but is not the only way possible.[5-8] It is really a means of generating test inputs. In theory, the system is optimal to all test inputs (and their variety can be made large indeed) that can be generated by initial conditions in a given command generator. In practice, one designs subjectively, as in the classical design, with one or two specific test inputs. The method differs from the classical approach in that for optimality the entire command generator's state is to be employed; however, most test imputs can be generated by first- or second-order systems. If the use of additional state variables is found to be worthwhile, then they should be measured, if it is feasible to do so, or be reconstructed by an observer.

III. Implicit Model-Following: Control Law

Since the implicit model performance index (7) is equivalent to that of (8), the treatment of which is well known (Ref. 1, p.47), there is no need for a separate derivation of the optimal control law. We have that

$$u=Kx, \qquad K=-\hat{R}^{-1}(\hat{S}+B^TP) \tag{34}$$

where P is the solution of the quadratic (Riccati) matrix equation

$$0=PA+A^TP-(PB+\hat{S}^T)\hat{R}^{-1}(\hat{S}+B^TP)+\hat{Q} \tag{35}$$

$$0=P(A-B\hat{R}^{-1}\hat{S})+(A-B\hat{R}^{-1}\hat{S})^TP-PB\hat{R}^{-1}B^TP+\hat{Q}-\hat{S}^T\hat{R}^{-1}\hat{S} \tag{36}$$

and where, repeated for convenience,

$$\hat{Q}=(CA-A_mC)^TQ_i(CA-A_mC), \hat{R}=R+B^TC^TQ_iCB,$$
$$\hat{S}=B^TC^TQ_i(CA-A_mC) \tag{37}$$

We separate, as in Ref. 3, K given by (34) into two components:

$$K=K_M+K_R \tag{38}$$

Here

$$K_M=-\hat{R}^{-1}\hat{S}=-(R+B^TC^TQ_iCB)^{-1}B^TC^TQ_i(CA-A_mC) \tag{39}$$

is the part due to $\hat{S}$ which was created by the implicit model, and

$$K_R=-\hat{R}^{-1}B^TP=-(R+B^TC^TQ_iCB)^{-1}B^TP \tag{40}$$

is the part affected by the Riccati Equation (35). In the case where (15) holds, $K_M=0$ and $K_R=-R^{-1}B^TP$, as in the standard case.

It is of interest to point out that K_M minimizes (in an ordinary, instantaneous sense) the integrand of the performance index (7). Letting $u=Kx$ in (1) and (7), the integrand w of (7) becomes

$$w=x^T[(CA_c-A_mC)^TQ_i(CA_c-A_mC)+K^TRK]x \tag{41}$$

where $A_c=A+BK$. By completing the square with respect to K, w can be put in the form

$$w=x^T[(K-K_M)^T(R+B^TC^TQ_iCB)(K-K_M)+\text{terms independent of } K]x \tag{42}$$

Since $(R+B^TC^TQ_iCB)>0$ and x is arbitrary, we have that the integrand of the performance index (7) is minimized over feedback matrices K by $K=K_M$, where K_M is given by (39).

Next we wish to study the effect on K of decreasing R (or, equivalently, of increasing Q_i) in the index (7), by defining

$$R=\rho R_o, \; R_o>0, \; \rho \text{ a positive scalar} \tag{43}$$

and letting $\rho\to 0$. We now assume that

$$B^TC^TQ_iCB>0 \tag{44}$$

K in now $K(\rho)$. It is evident from (8) and (9) that, instead of $K(\rho)$ increasing with $\rho\to 0$ as in the standard case (1-3), now $K(\rho)$ tends to some finite K_0, because the index is independent of ρ in the limit. This is quite reasonable, as the index (7) attempts to produce a "best" match between $A+BK$ and A_m rather than a "fastest" $A+BK$.

Suppose that exact model-matching is possible. By this we mean that there exists a $K=K_{\text{match}}$ so that $y(t)$, the solution of (4) and (6), satisfies $\dot{y}=A_my$; this leads to

$$CA+CBK_{\text{match}}=A_mC \tag{45}$$

Clearly, now $K_0=K_{\text{match}}$, and, since the integrand of (7) with $R=0$ is now minimized (to zero), we have by the previous result that $K=K_M$. Hence,

$$\lim_{\rho\to 0} K_M(\rho)=K_{\text{match}}, \qquad \lim_{\rho\to 0} K_R(\rho)=0 \tag{46}$$

To prove these assertions, we let $\rho=0$ and use (45). Then, in (39),

$$K_M=-(B^TC^TQ_iCB)^{-1}B^TC^TQ_i(-CBK_{\text{match}})=K_{\text{match}}$$

and in (40) $P=0$ because, in (36), the forcing term

$$\hat{Q}-\hat{S}^T\hat{R}^{-1}\hat{S}=(CA-A_mC)^T$$
$$[I-CB(B^TC^TQ_iCB)^{-1}B^TC^TQ_i](CA-A_mC)$$

is zero if (45) holds [or if CB is nonsingular, in which case (45) always holds]. To summarize, *assuming* (43) *and* (44) *and letting* $\rho\to 0$ *results in* $K\to K_0$; *if exact model-matching* (45) *is possible then* $K_R\to 0$ *and* $K_M\to K_{\text{match}}$.

Lastly, what kind of control laws can be achieved by using implicit model-following? The answer is that *every desired control law* $u=K_dx$ *can be obtained by minimizing the performance index* (7) *with a suitable model matrix* A_m. To see this, set in (7)

$$C=I,\ R=0,\ A_m=A+BK_d \tag{47}$$

Then the integrand of I_i in (7) becomes

$$(\dot{x}-A_mx)^TQ_i(\dot{x}-A_mx)=(Bu-BK_dx)^TQ_i(Bu-BK_dx)$$
$$=(u-K_dx)^TB^TQ_iB(u-K_dx)$$

so that I_i is minimized if and only if $B(u-K_dx)=0$, or if $u=K_dx+z$, for any vector z such that $Bz=0$. (This nonuniqueness of u has no effect and disappears if $B^TQ_iB>0$.)

In practice, it may not be easy to find A_m to achieve a specific K_d, e.g., an A_m such that certain elements of K are zero and the others achieve desired closed-loop properties. If a destabilizing K_d is desired (as in Sec. V), it may be necessary to employ a factor $e^{-2\alpha}$ in the integrand of I_i, with a sufficiently large $\alpha>0$ to insure the convergence of the integral I_i.[10] By augmenting the plant's state x_a with the state x_c of a command generator, the desired control law will include a feedforward controller as well as a feedback one.

$$u=K_dx=K_{fb}x_a+K_{ff}x_c \tag{48}$$

Thus, the preceding result covers feedforward controllers, and, furthermore, it can be extended, along the lines of Ref. 10, to include dynamic feedback and feedforward controllers as well. The result does not hold and the proof does not go through, of course, for the cases discussed in the previous section where the cross-product term vanishes; we have then the standard index (2), which results in a control law that necessarily has very special and desirable properties.[1]

The question of which models can be matched exactly by a control law (48) is a separate problem[4,11,12,] the investigation of which leads to algebraic conditions. Important as these conditions are, they often cannot be met or may require excessive control magnitudes. An advantage of optimal implicit model-following is that is provides, relatively easily, a subjective balance between the desirable and the possible. And we have shown here that, if the model can be matched exactly and the price in terms of controls is not too high (by letting $R\to 0$), then the desirable is achieved automatically.

IV. Comparison of Methods: Discussion

The explicit model configuration (Fig. 1) is conspicuous by the presence of the simulation of the model; the implicit model method omits the model and K_4 blocks in Fig. 1. This extra complexity of the explicit model controller should not be considered a serious drawback these days if the method offers performance advantages. Another, less obvious inherent feature of the explicit model method is that the model's equations affect only the feedforward gain matrices K_3 and K_4 but not the feedback matrices K_1 and K_2. This can be shown by an appropriate partitioning of the Riccati equation,[3] and it can be explained as follows: suppose that the initial conditions are only on the state of the plant, but zero for the command generator and model. Then, in effect, the model does not exist, and the feedback matrices K_1 and K_2 are determined only by the choice of Q_e and R in the index I_e of (10). This feature might be disadvantageous if the plant is excited by disturbances (say wind gusts), which of course do not affect the explicit model. Explicit model-following then may behave poorly, whereas the implicit model system, if the desirable response to these disturbances in incorporated in the model, can respond better, since it attempts direct model-matching (including model pole-matching, which explicit model-following does not provide).

Since the model-following property of the configuration of Fig. 1 depends on the command signal first passing through the explicit model in the feedforward path, the system of Fig. 1 can be expected to be slower to respond than that of implicit model-following which is inherently closer to the model. To achieve a comparable degree of model-following, the gains K_1 and K_2 in Fig. 1 can be expected to be higher than those obtained via implicit model-following, because model-following in Fig. 1 requires a "fast" plant. On the other hand, the explicit model method can be expected to produce better steady-state performance, since it attempts to reduce the error between the actual and model states, whereas the index I_i of (7) attempts, roughly speaking, to reduce the error between the derivatives. Also, intuitively, one expects the sensitivity to plant parameter variations to be lower in the configuration of Fig. 1, because the explicit model will act to reduce the resulting errors.

A potential advantage of the implicit model method is that less trial-and-error is required to achieve a balanced design, since the implicit model in effect defines (a nondiagonal) Q in the performance index.

One can attempt to combine the advantages of both methods by minimizing

$$I=\int_0^\infty[(\dot{y}-A_my)^TQ_i(\dot{y}-A_my)$$
$$+(y-z)^TQ_e(y-z)+u^TRu]\,dt \tag{49}$$

This results in a configuration of Fig. 1, but now the feedbacks K_1 and K_2 depend on the model. We did not experiment with this performance index.

Both methods have the useful feature that, once the controls needed to emulate the model have been achieved, there is no reason for their magnitude to increase with Q, as with the standard index. Thus, by specifying a model that is not too far above the capabilities of the closed-loop system, we insure that the controls will not exceed their bounds.

V. Comparison of Methods: Lateral-Axis Control

The lateral-axis linearized equations for an F-4 aircraft, as given in Ref. 7 for flight condition 2 (Mach. 0.5, altitude 5,000 ft), are

$$\dot{x}_a=\begin{bmatrix}p\\r\\\beta\\\phi\end{bmatrix}^{\!\cdot}=\begin{bmatrix}-1.768 & 0.4125 & -14.25 & 0\\ -0.007 & -0.3831 & 6.038 & 0\\ 0.0016 & -0.9975 & -0.1551 & 0.0586\\ 1 & 0 & 0 & 0\end{bmatrix}$$
$$\times\begin{bmatrix}p\\r\\\beta\\\phi\end{bmatrix}+\begin{bmatrix}1.744 & 8.952\\ -2.92 & -0.3075\\ 0.0243 & -0.0036\\ 0 & 0\end{bmatrix}\begin{bmatrix}\delta_r\\\delta_a\end{bmatrix}=A_ax_a+B_ax_\delta \tag{50}$$

where the state variables p, r, β, and ϕ are roll rate (rad/sec), yaw rate (rad/sec), sideslip (rad), and bank angle (rad), respectively. The actuator dynamics for the rudder and aileron deflections δ_r (rudder) and δ_a (aileron) are described by

$$\dot{x}_\delta = \begin{bmatrix} \dot{\delta}_r \\ \dot{\delta}_a \end{bmatrix} = \begin{bmatrix} -20 & 0 \\ 0 & -10 \end{bmatrix} \begin{bmatrix} \delta_r \\ \delta_a \end{bmatrix} + \begin{bmatrix} 20 & 0 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} u_r \\ u_a \end{bmatrix} = A_\delta x_\delta + B_\delta u \tag{51}$$

The model equations are

$$\dot{x}_m = \begin{bmatrix} \dot{p}_m \\ \dot{r}_m \\ \dot{\beta}_m \\ \dot{\phi}_m \end{bmatrix} = \begin{bmatrix} -4 & 0.865 & -10 & 0 \\ .04 & -0.507 & 5.87 & 0 \\ 0 & -1 & -0.743 & 0.0586 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_m \\ r_m \\ \beta_m \\ \phi_m \end{bmatrix} + \begin{bmatrix} 3.3 & 20 \\ -3.13 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_{cr} \\ \delta_{ca} \end{bmatrix} A'_a x_m + B'_a \delta_c \tag{52}$$

where the parameters were selected to provide desirable handling qualities.[7]

The pilot's rudder and aileron commands, δ_{cr} and δ_{ca}, are assumed to be pulses shown in Fig. 2, $\delta_c(t) = ke^{-\omega t}\sin\omega t$, the solution of

$$\ddot{\delta}_c + 2\omega\dot{\delta}_c + 2\omega^2\delta_c = 0, \quad \dot{\delta}_c(0) = k\omega \tag{53}$$

The coordinates (t_p, s_p) of the pulse's peak can be selected by specifying the parameters k and ω:

$$k = (e^{\pi/4}s_p)/(\sin\pi/4), \qquad \omega = \pi/4t_p \tag{54}$$

The command generator's equations are, thus,

$$\dot{x}_c = \begin{bmatrix} \dot{\delta}_{cr} \\ \ddot{\delta}_{cr} \\ \dot{\delta}_{ca} \\ \ddot{\delta}_{ca} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2\omega_r^2 & -2\omega_r & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -2\omega_a^2 & -2\omega_a \end{bmatrix} \begin{bmatrix} \delta_{cr} \\ \dot{\delta}_{cr} \\ \delta_{ca} \\ \dot{\delta}_{ca} \end{bmatrix} = A_c x_c \tag{55}$$

$$\delta_c = \begin{bmatrix} \delta_{cr} \\ \delta_{cr} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x_c = C_c x_c \tag{56}$$

where the subscripts r and a denote rudder and aileron, respectively.

The objective is to achieve a response $x_a(t)$ to a pilot aileron command [lateral stick $\delta_{ca}(t)$ with $s_p = 0.2$ rad, $t_p = 0.5$ sec], so that $x_a(t)$ will be as close as possible to the response $x_m(t)$ of the model, within constraints on the control surface deflections and deflection rates. This is to be done for both the implicit and explicit model-following, thus providing a comparison.

Implicit Model-Following

This is the case with $CB = 0$ discussed in Sec. II; the model A_m then serves to define a nondiagonal $\hat{Q}$ according to (28). Furthermore, the fourth through eighth elements of $\dot{y} - A_m y$ are zero, so that, if we consider a diagonal Q_i, only its first three elements affect the index. Thus,

$$Q_i = \begin{bmatrix} Q_I & 0 \\ 0 & 0 \end{bmatrix}, \quad Q_I = \text{diag}[q_1 \; q_2 \; q_3], \quad R = I \tag{57}$$

The augmented 10×10 plant matrix A is given by (23, 50, 51, and 55), and the augmented 8×8 model matrix A_m is given by (25) and (52).

We experimented with $Q_I = \alpha I$, increasing α by factors of 10 from $\alpha = 0.001$. Beyond $\alpha = 0.1$, the changes in response and pole placement were small; most of the gains, however, continued to increase roughly as $\alpha^{1/2}$, and the actuator poles tended to $-\infty$. A much better response in $r(t)$ and $\beta(t)$ was provided by $Q_I = \text{diag}[0.1, 10, 0.1]$.

The results for the gains, the maximal magnitudes of the control surface angles and rates, and the closed-loop eigenvalues are given in Tables 1-3. The responses of the state variables are shown in Fig. 3. It is interesting to note that (here, at least) the implicit model method provides very good pole-matching. However, pole-matching is not the objective; indeed, the Dutch roll poles start to diverge after $\alpha = 0.1$. The unstable desired spiral mode pole could be approached by multiplying the integrand of the performance index by $e^{-2\gamma t}$, $\gamma > 1.75 \times 10^{-5}$; however, we had no opportunity to employ this device.

The feedback gains K_2 around the actuators tend to speed up their response. Since we must assume that the actuators' response cannot be improved further by feedback, we minimized[13] the performance index also with the constraint $K_2 = 0$. The results are indicated in footnote ***a*** in the tables and shown in Fig. 3; naturally, there is a slight deterioration in response.

Note that the entire command generator's state is fed forward; thus, we have the need to measure the rates of the commands. In fact, the aileron's angle $\delta_a(t)$ is very similar to the command pulse $\delta_{ca}(t)$, and because of the rate feedforward, it is leading slightly. Its peak coordinates are $\tau_p = 0.35$ sec, $s_p = 0.28$ rad vs $\tau_p = 0.5$ sec, $s_p = 0.2$ rad of the command pulse.

Explicit Model-Following and Comparison of Methods

Here we minimize

$$I_e \int_0^\infty [(x_a - x_m)^T Q_e (x_a - x_m) + u^T R u] dt \tag{58}$$

subject to the fourteenth-order augmented plant equation:

$$\dot{x} = \begin{bmatrix} \dot{x}_a \\ \dot{x}_\delta \\ \dot{x}_m \\ \dot{x}_c \end{bmatrix} = \begin{bmatrix} A_a & B_a & \multicolumn{2}{c}{0} \\ 0 & A_\delta & & \\ \multicolumn{2}{c}{0} & A_m & B_m \\ & & 0 & A_c \end{bmatrix} \begin{bmatrix} x_a \\ x_\delta \\ x_m \\ x_c \end{bmatrix} + \begin{bmatrix} 0 \\ B_\delta \\ 0 \\ 0 \end{bmatrix} u \tag{59}$$

In our experiments, we selected $R = I$ and $Q_e = \alpha I_{4\times 4}$, where α was increased from $\alpha = 0.01$ by multiples of 10. The resulting gains and the maximum control surface angles and rates are given in Tables 1 and 2.

Table 1 Gains for lateral-axis control

Q	Feedback gains K_1 p	r	β	ϕ	K_2 δ_r	δ_a
Implicit model method						
$0.001\ I$	−0.00053	0.077	0.0016	−0.0044	−0.017	−0.0110
	−0.024	−0.016	0.11	0.00067	−0.0056	−0.060
$0.1\ I$	−0.15	0.29	0.27	−0.01	−0.47	−0.60
	−0.63	0.003	1.70	0.002	−0.30	−2.10
$10.0\ I$	−1.60	0.85	3.50	−0.002	−9.30	−6.90
	−6.8	1.20	14.30	−0.003	−3.50	−26.7
[0.1,10,0.1]	−0.16	0.44	0.53	−0.002	−8.30	−0.99
	−0.65	−0.045	1.76	0.0001	−0.49	−2.20
[0.1,10,0.1] [a]	−0.024	0.018	−0.043	0.001	0	0
	−0.45	−0.154	−0.234	0.012	0	0
Explicit model method						
$0.01\ I$	0.026	0.170	−0.043	0.037	−0.022	0.020
	−0.57	−0.106	0.159	−0.079	0.010	−0.046
$1.0\ I$	−0.082	1.06	−0.107	−0.092	−0.150	−0.042
	−0.875	−0.403	2.06	−0.980	−0.021	−0.595
$1.0\ I$ [a]	−0.067	1.00	−0.194	−0.077	0	0
	−0.776	−0.443	1.79	−0.781	0	0
$100.0\ I$	−1.86	9.94	−4.49	−2.14	−1.05	−0.474
	−9.46	−2.25	7.10	−9.77	−0.237	−0.319

[a] Zero actuator-feedback constraint.

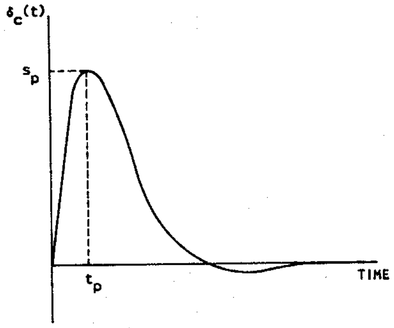

Fig. 2 Typical pilot command pulse.

As in the implicit model design, the maximal control surface deflections do not increase monotonically with α. Thus, we adopted the level of gains as the basis of comparison. For $Q_e = I$, the gains are of the same general level (or slightly higher) as those for the case (57), with $Q_I = \text{diag}\ [0.1,10,0.1]$ in the implicit model design. The levels of maximum surface deflection and deflection rates are also comparable for these two cases. The responses of $x_a(t)$ for $Q_e = I$ are shown in Fig. 3. Evidently, the response of yaw rate and sideslip are better in the implicit model design, whereas for bank angle the implicit model response has a steady-state error. This error is of

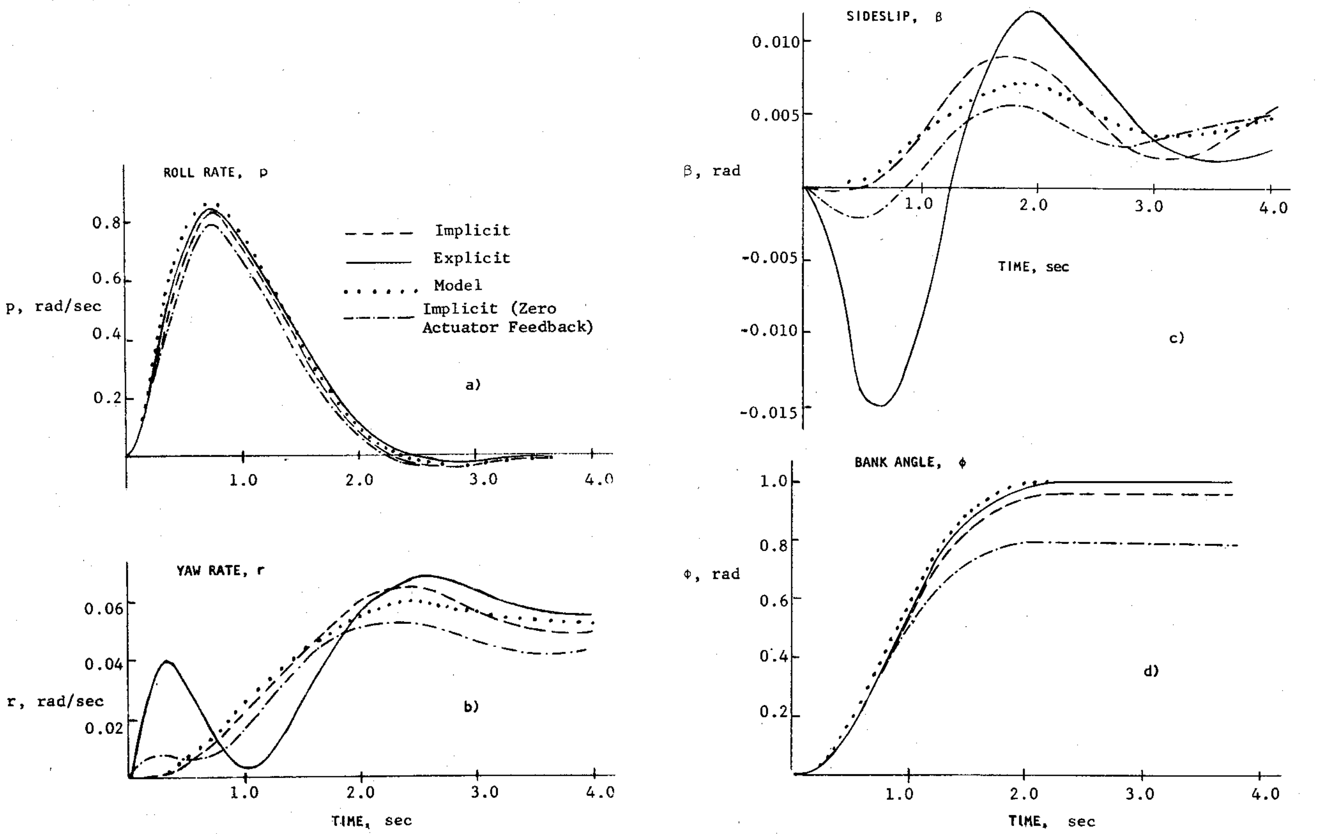

Fig. 3 Implicit and explicit model responses with comparable feed back gain levels: a) rp;; rate. b) yaw rate, c) sideslip, and d) bank angle.

Table 1 (continued from opposite page)

Feedforward gains								
K_3				K_4				
δ_{cr}	$\dot{\delta}_{cr}$	δ_{ca}	$\dot{\delta}_{ca}$	p_m	r_m	β_m	ϕ_m	
−0.0026	−0.0071	−0.0270	−0.0270	...	...	...	...	u_r
0.047	0.0070	0.260	0.032	...	...	...	...	u_a
0.77	0.01	1.10	−0.09	...	...	...	...	u_r
0.91	0.03	6.0	0.22	...	...	...	...	u_a
11.70	0.07	13.40	−0.02	...	...	...	...	u_r
8.10	0.01	62.0	0.22	...	...	...	...	u_a
9.80	0.056	0.055	0.026	...	...	...	...	u_r
0.97	0.016	6.20	0.21	...	...	...	...	u_a
1.06	0.042	−0.096	−0.036	...	...	...	...	u_r
0.46	−0.027	2.65	0.045	...	...	...	...	u_a
0.039	−0.00003	−0.211	−0.064	−0.012	−0.057	−0.014	−0.045	u_r
−0.016	−0.009	−0.325	0.081	0.031	0.044	0.047	0.086	u_a
0.262	−0.088	−0.309	−0.138	0.088	−0.746	−0.124	0.072	u_r
0.268	0.027	2.61	0.279	0.634	0.283	−0.957	0.990	u_a
0.221	−0.088	−0.302	−0.138	0.079	−0.687	−0.086	0.057	u_r
0.195	0.027	1.90	0.279	0.551	0.262	−0.893	0.794	u_a
0.754	−0.413	0.520	−0.049	1.75	−9.19	3.79	2.08	u_r
1.05	0.077	8.95	0.249	8.52	2.25	−4.48	9.79	u_a

Table 2 Maximal control surface deflections

Q	δ_r, ± rad	δ_a, ± rad	$\dot{\delta}_r$, ± rad/sec	$\dot{\delta}_a$, ± rad/sec
Implicit model method				
0.00 I	0.019	0.045	0.344	0.250
0.1 I	0.049	0.270	0.613	1.75
10.0 I	0.039	0.270	0.194	1.79
diag [0.1,10,0.1]	0.039	0.280	0.200	1.690
diag [0.1,10,0.1] [a]	0.035	0.300	0.344	1.25
Magnitude constraint	0.262	0.524	0.437	1.12
Explicit model method				
0.01 I	0.056	0.079	0.613	0.550
1.0 I	0.091	0.264	1.150	1.940
1.0 I [a]	0.096	0.280	1.20	2.1
100.0 I	0.043	0.294	0.313	1.81

[a] Zero actuator-feedback constraint.

Table 3 Closed-loop eigenvalues for lateral-axis control implicit model method

Q_I	$\lambda_{1,2}$ Actuators	$\lambda_{3,4}$ Dutch roll	λ_5 Basic roll	λ_6 Spiral mode
0.001 I	−20.1, −10.3	−0.337±j2.48	−2.095	-1.37×10^{-2}
0.1 I	−34.6, −23.5	−0.561±j2.40	−3.885	$-5{,}897\times10^{-3}$
10.0 I	−317.0, −163.0	−0.354±j2.42	−4.009	-1.736×10^{-3}
100.0 I	−1002.0 −511.0	−0.348±j2.42	−4.011	-1.54×10^{-3}
diag [0.1,10,0.1]	−186.0, −29.2	−0.332±j2.42	−3,848	-4.362×10^{-3}
diag [0.1,10,0.1] [a]	−19.9, ...	−0.321±j2.42	−5.88±j4.77	-4.13×10^{-3}
Desired model	... ...	−0.616±j2.42	−4.017	$1.75\times10^{/5}$

[a] Zero actuator-feedback constraint. Note that two previously real eigenvalues became a complex pair.

no consequence in this application and could be eliminated by adding a term

$$q(\phi_a-\phi_m)^2, \qquad q>0$$

im the implicit model method index I_i of (7), resulting in an index of the type (49). [There may be no need to simulate the entire model to obtain ϕ_m: we observe that $p_m(t)=\dot{\phi}_m(t)$ is similar to the command $\delta_{ca}(t)$; thus integration of $\delta_{ca}(t)$ and a gain factor may give a good approximation of $\phi_m(t)$ in the steady state.]

The results for $Q_e=I$ with the constraint $K_2=0$ (no actuator feedback) are tabulated but not shown in Fig. 3 because the change in response in negligible. If roughly tenfold higher feedback gains are allowed, as for $\alpha=100$, then the error in explicit model-following is very small. With the explicit model method, there is no pole-matching (because the feedback K_I is independent of model); the eigenvalues for the case $Q_e=I$ are (except those for the model and command generator, which are unchanged)

$$\lambda_{1,2}=-1.81\pm j1.99, \qquad \lambda_3=-0.99,$$

$$\lambda_{4,5}=-8.4\pm j4.51, \qquad \lambda_6=-19.7$$

As a result, responses for initial conditions in the main plant (initial rollrate; see Ref. 15) were quite different from those of the model, whereas in the implicit model design there is good model-following in $p(t)$ and $\phi(t)$, and fair model-following in $r(t)$ and $\beta(t)$. Of course, it was assumed, that, although the aircraft is subject to the initial condition causing disturbance, the explicit model in the feedforward path is unaffected and remains at rest.

We made a sensitivity comparison by assuming a flight condition change from the nominal Mach 0.5, 5,000-ft alt to Mach 2.15, 45,000-ft alt; this reduces roughly by half many of the entries of A_a and B_a. In both designs, the model-following was still good in roll rate and bank angle, and deteriorated in yaw rate and sideslip. On balance, the change from nominal response was negligibly larger in the implicit model design.[14] We ran both designs, with the gains computed for the nominal aileron command pulse with a faster pulse, one with $t_p = 0.2$ sec (down from $t_p = 0.5$ sec); although in both designs the sensitivity to the change was small, the implicit model method provided somewhat better model-following.[14]

VI. Conclusions

The implicit model method of linear feedback control system design via linear-quadratic optimization, explored in some detail in this paper, has the attractive feature that it defines a quadratic performance index so that the closed-loop dynamics approach that of a desired model. In this objective, it performed, for our application to lateral-axis flight control, better than explicit model-following, in particular in regard to the initial transient. We successfully applied implicit model-following to the problem of decoupling the modes of a VTOL aircraft in low speed.[15]

The advantages of explicit model-following were slight: lower feedback gains around the actuators and slightly lower sensitivity to plant parameter changes. Thus, on balance, implicit model-following appears to have the upper hand. Needless to say, the eyeballing comparison made here is subjective and certainly not conclusive. In particular, the potential sensitivity superiority of the explicit model method should be explored in particular applications; it was disappointingly negligible in our case. Of course, should the explicit model method prove to have advantages, the two methods can be combined as suggested in Sec. IV.

A work of caution: care should be exercised in the specification of the desired model for implicit model-following. Good results can be expected if the model puts reasonable demands on the closed-loop system; the method failed in an experiment with an unrealistic model. This points to a direction of research: the translation of design specifications and desiderata into compatible model equations.

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